A two-sample test for comparison of long memory parameters

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Mots clés :

1. Processus
2. Statistique mathématique

1 Introduction

We propose a procedure for testing the null hypothesis $d_1 = d_2$ that long memory parameters $d_i \in [0, 1/2)$ of two samples of length n, taken from respective stationary processes X_i, $i = 1, 2$, are equal, against the alternative $d_1 > d_2$. A natural extension of one-sample test developed in [1] about unknown long memory parameter d is to consider test statistic, T_n of the form

$$T_n = \frac{V_1/S_{11,q}}{V_2/S_{22,q}} + \frac{V_2/S_{22,q}}{V_1/S_{11,q}}, \tag{1}$$

where $V_i/S_{ii,q}$ is computed from sample $(X_i(1), \dots, X_i(n))$ $(i = 1, 2)$. Here, V_i is the empirical variance of partial sums of X_i

$$V_i = n^{-2} \sum_{k=1}^{n} \left(\sum_{t=1}^{k} (X_i(t) - \bar{X}_i) \right)^2 - n^{-3} \left(\sum_{k=1}^{n} \sum_{t=1}^{k} (X_i(t) - \bar{X}_i) \right)^2. \tag{2}$$

$S_{ii,q}$ is the Newey-West or HAC estimator of the long-run variance of X_i (see [3])

$$S_{ij,q} = \sum_{h=-q}^{q} \left(1 - \frac{|h|}{q+1} \right) \hat{\gamma}_{ij}(h), \tag{3}$$

where $\gamma_{ij}(h)$ are the empirical cross covariances of samples $(X_1(1), \dots, X_1(n))$ and $(X_2(1), \dots, X_2(n))$.

From [1], one easily derives the asymptotic null distribution T of the statistic T_n under the condition that the two samples are independent. It is also easy to show that for $d_1 \neq d_2$, one of the ratios in (1) tends to infinity and the other one to zero, meaning that the test is consistent against the alternative $d_1 \neq d_2$.

However, independence of the two samples is too restrictive and may be unrealistic in many applications. In order to eliminate the eventual dependence between samples, a modification $\tilde{T}_n$ of (1) is proposed, which uses residual observations $(\tilde{X}_1(1), \ldots, \tilde{X}_1(n))$, obtained by regressing partial sums of X_1 on partial sums of X_2.

$$\tilde{X}_1(t) \;=\; X_1(t) - (S_{12,q}/S_{22,q})X_2(t), \qquad t = 1, \ldots, n. \tag{4}$$

The statistics $\tilde{T}_n$ is defined as follows

$$\tilde{T}_n \;=\; \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} + \frac{V_2/S_{22,q}}{\tilde{V}_1/\tilde{S}_{11,q}}. \tag{5}$$

$\tilde{V}_1, \tilde{S}_{11,q}$ are the statistics in (2), (3), respectively, where $X_1(t), t = 1, \cdots, n$ is replaced by $\tilde{X}_1(t), t = 1, \cdots, n$.

2 Asymptotic properties of $\tilde{T}_n$

Consider the class of bivariate linear models $((X_1(t), X_2(t)),\ t \in \mathbf{Z})$ given by

$$X_i(t) \;=\; \sum_{k=0}^{\infty} \psi_{i1}(k)\xi_1(t-k) + \sum_{k=0}^{\infty} \psi_{i2}(k)\xi_2(t-k), \qquad i = 1, 2, \tag{6}$$

where

- $\psi_{ij}(k) \sim |k|^{d_{ij}-1} \quad (k \to \infty)$, and $d_{ij} \in (0, 1/2)$.
- $(\xi_1(t), \xi_2(t))$, $t \in \mathbf{Z}$ is a bivariate (weak) white noise with nondegenerate covariance matrix
$$\begin{pmatrix} 1 & \rho_\xi \\ \rho_\xi & 1 \end{pmatrix}. \tag{7}$$

Let $q = q_n \to \infty$ and $n/q_n \to \infty$ as $n \to \infty$.
Under these conditions, the statistic $\tilde{T}_n$ satisfies the following convergence results :

(i) If $d_1 = d_2 = d$ then

$$\tilde{T}_n \;\to_{\text{law}}\; \tilde{T}, \tag{8}$$

where the distribution of $\tilde{T}$ only depends on $(d1, d2)$.

(ii) If $d_1 > d_2$ then

$$\tilde{T}_n \;\to_p\; \infty. \tag{9}$$

3 Testing procedure

Let $t_\alpha(d)$ denote the upper $\alpha-$quantile of the r.v. $\tilde{T}$ defined in (8) when $\rho = 0$), viz.

$$\alpha = P(\tilde{T} > t_\alpha(d)), \qquad \alpha \in (0,1). \tag{10}$$

Let

$$\hat{d} = (\hat{d}_1 + \hat{d}_2)/2, \tag{11}$$

where $\hat{d}_i$ is an estimator of d_i satisfying

$$\hat{d}_i - d_i = o_p(1/\log n) \qquad (i = 1, 2). \tag{12}$$

Similarly to [1], it can be proved that the quantile function $t_\alpha(d)$ is continuous in $d \in [0, 1/2)$ for any $\alpha \in (0,1)$. Therefore, the estimated quantile $t_\alpha(\hat{d}) \to_p t_\alpha(d)$ as $n \to \infty$ and the asymptotic level of the tests associated to the critical regions in (13) is preserved by replacing $t_\alpha(d)$ by $t_\alpha(\hat{d})$.

Testing the equality of the memory parameters in the case of possibly dependent samples. We wish to test the null hypothesis $d_1 = d_2$ against the alternative $d_1 > d_2$ in the general case when X_1 and X_2 are possibly dependent. The decision rule at $\alpha-$level of significance is the following: we reject the null hypothesis when

$$\tilde{T}_n > t_\alpha(\hat{d}). \tag{13}$$

The consistency of this test is ensured by the result given in section 2.

Bibliographie

[1] Giraitis L., Leipus, R., Philippe, A., 2006. A test for stationarity versus trends and unit roots for a wide class of dependent errors. Econometric Theory 21, 989–1029.
[2] Lavancier, F., Philippe, A., Surgailis, D., 2009. Covariance function of vector self-similar process. Statist. Probab. Letters 79, 2415–2421.
[3] Abadir, K., Distaso, W., Giraitis, L., 2009. Two estimators of the long-run variance: beyond short memory. Journal of Econometrics 150, 56–70.